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Approximation by Gradients

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1. INTRODUCTION

Approximate Cauchy problem for the Laplace operator Δ

Let σ be a compact oriented C^1-hypersurface in $\mathbb{R}^n$ with C^1 boundary if it has one.

A. Given any pair of continuous functions f, g on σ and $\epsilon > 0$ there exists a function h harmonic in a neighborhood of σ such that

$$|h - f| + \left|\frac{\partial h}{\partial n} - g\right| < \epsilon$$

on σ where $\partial/\partial n$ denotes normal derivation, taking the unit normals to vary on only one side of the surface σ.

This is proved by Mergelyan [1] in case σ is the image of the unit disc in $\mathbb{R}^3$. (His proof would work out even for higher dimensions if σ is homeomorphic to a spherical cap.)

B. Here we consider a related problem, an answer to which would include A. Let E be an arbitrary compact set in $\mathbb{R}^n$ and let $u \in C^1(\mathbb{R}^n)$ and $\epsilon > 0$. Under what kind of restrictions on u and E does there exist a function h, harmonic in a neighborhood of E, such that $|\nabla u - \nabla h| < \epsilon$ on E where ∇ is the traditional notation for the gradient?

Further, we ask whether it is possible to replace ∇u by arbitrary continuous mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus we pose the following problem.

C. Let E be an arbitrary compact set in $\mathbb{R}^n$. Under what conditions on E can it be guaranteed that given any $\epsilon > 0$ and an arbitrary continuous function $V: \mathbb{R}^n \to \mathbb{R}^n$, there exists a $u \in C^1(\mathbb{R}^n)$ such that $|V - \nabla u| < \epsilon$ on E?

Our answers to B and C are far from complete. In answer to B we prove in Section 2 the following:

THEOREM 1. *If* meas$(E) = 0$, *then for every* $\epsilon > 0$ *and every* $u \in C^1(\mathbb{R}^n)^2$ *there exists a function h, harmonic in a neighborhood of E (both h and the neighborhood depend on* ϵ *and u), such that* $|\nabla(u - h)| < \epsilon$ *on E.*

From this theorem we deduce *A*. In Section 3 we answer C in the following cases: (i) E is totally disconnected; (ii) E is the support of a Jordan arc; and (iii) E is the support of a nonrectifiable Jordan curve.

In the case where E is a rectifiable Jordan curve, naturally we cannot approximate arbitrary continuous functions by gradients, since gradients have their integrals zero along the curve while mere continuous functions need not. But we prove the best possible: Let γ be our curve and let $V: \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Further assume $\int_\gamma V \cdot dx = 0$. Then V can be approximated by ∇u, $u \in C^1(\mathbb{R}^n)$.

2. *Proof of Theorem* 1. We may, without loss of generality, assume $u \in C_0^\infty(\mathbb{R}^n)$ (infinitely differentiable and with compact support). Then it is a standard formula (let us work in $\mathbb{R}^3$ for simplicity, proof for other dimensions being similar)

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta u(y)}{|x-y|}\, dy.$$

Let us take any $\delta > 0$ and $E_\delta = \{x; |x - y| < \delta \text{ for some } y \in E\}$, δ-neighborhood of E;

$$h_\delta(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{-E_\delta} \frac{\Delta u(y)}{|x-y|}\, dy,$$

and

$$K_\delta(x) = -\frac{1}{4\pi}\int_{E_\delta} \Delta u(y)\, \nabla_x \frac{1}{|x-y|}\, dy$$

where ∇_x denotes gradient with respect to the variable x. We now assert that $\nabla u(x) = \nabla h_\delta(x) + K_\delta(x)$, $\forall x \in E_\delta$. It is clear that

$$\nabla u = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta\, \nabla u(y)}{|x-y|}\, dy = \frac{1}{4\pi}\int_{\mathbb{R}^3} \Delta u(y)\, \nabla_y \frac{1}{|x-y|}\, dy =$$

(an application of Fubini and derivative of a product formula)

$$= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \Delta u(y)\, \nabla_x \frac{1}{|x-y|}\, dy$$

and hence

$$\nabla u(x) = K_\delta(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3 - E_\delta} \Delta u(y) \nabla_x \frac{1}{|x-y|} dy.$$

For $x \in E_\delta$, ∇_x in the second term of R.H.S. can be brought outside the integral and so we have

$$\nabla u(x) = \nabla h_\delta(x) + K_\delta(x) \quad \text{on} \quad E_\delta.$$

We shall now prove that $\max_{E_\delta} |K_\delta| = M_\delta \to 0$ as $\delta \to 0$. Let $M = \sup |\Delta u|$. Then $M_\delta \leqslant C \cdot M(\text{meas } E_\delta)^{1/3}$ where C is an absolute constant. It is clear that

$$|K_\delta(x)| \leqslant \frac{1}{4\pi} \int_{B(x,R)\cap E_\delta} |\Delta u(y)| \left| \nabla_x \frac{1}{|x-y|} \right| dy$$

$$+ \frac{1}{4\pi} \int_{E_\delta - B(x,R)} |\Delta u(y)| \left| \nabla_x \frac{1}{|x-y|} \right|$$

$$\leqslant M \cdot R + \frac{M}{4\pi R^2} \text{meas } (E_\delta)$$

for all $R > 0$. Select $R = (\text{meas } E_\delta)^{1/3}$, then $|K_\delta(x)| \leqslant (1 + 1/4\pi) \cdot M \cdot (\text{meas } E_\delta)^{1/3}$. Q.E.D.

Deduction of A from Theorem 1

From the hypothesis of A, it is immediate that meas $\sigma = 0$ and by our Theorem 1, it would follow that given any $u \in C^1(\mathbb{R}^n)$, u and $\partial u/\partial n$ can be approximated by H and $\partial H/\partial n$, respectively, and simultaneously on σ where H is harmonic in a neighborhood of σ.

Thus we are reduced to proving that given any $\epsilon > 0$ and any pair of continuous functions f, g on σ, there exists a $u \in C^1(\mathbb{R}^n)$ such that $|u - f| + |\partial u/\partial n - g| < \epsilon$ on σ.

At this stage we may assume $f = 0$. For every $x \in \sigma$, there exists a neighborhood V_x and a $U_x \in C^1(\mathbb{R}^n)$ such that $u_x = 0$ on $V_x \cap \sigma$ and $|\partial u_x/\partial n - g| < \epsilon$ on $V_x \cap \sigma$. Assuming we have proved this, we can select out of these V_x a finite number, for example, V_i $(1 \leqslant i \leqslant m)$ covering σ and let $\{\varphi_i\}$ be a partition of unity subordinate to $\{V_i\}$. Then we set $u = \sum_{i=1}^m \varphi_i u_i$. Then $u = 0$ on σ and $|\partial u/\partial n - g| = |\sum_{i=1}^m \varphi_i(\partial u_i/\partial n)| < \epsilon$ on σ. Now we shall prove the local version. Given a point on σ, we can assume that there exists a neighborhood of that point in which σ would look like $z = \varphi(x, y)$

where φ is C^1 (here for simplicity we assume $n = 3$). Let $u = (z - \varphi) \cdot g_1$ where $g_1 \in C^1(\mathbb{R}^n)$. Then $u = 0$ on σ and

$$\frac{\partial u}{\partial n} = \sqrt{\varphi_x^2 + \varphi_y^2 + 1} \cdot g_1$$

on σ. Hence if we select g_1 such that

$$|(\varphi_x^2 + \varphi_y^2 + 1)^{1/2} g_1 - g| < \epsilon,$$

we are finished. Q.E.D.

3. Theorems 3 and 4, proved in this section, were first proved in $\mathbb{R}^2$ using complex variable methods. The fact that nonrectifiability allows for approximation but not rectifiability was formerly an intriguing one, since one could not readily see it in higher dimensions and the complex variable proof was far from transparent. Now that we have proofs for all dimensions which are more natural, some of the surprise is lost, but we feel that the proof in $\mathbb{R}^2$ should still be given, at least for its elegance.

Let C be a nonrectifiable Jordan curve. We may assume the origin is in the interior of C. Then any complex-valued continuous function can be approximated by $P(z) + Q(1/z)$ where P, Q are polynomials in one variable. Except for the $1/z$ part, the rest certainly possesses a primitive in a neighborhood of C and so we need only prove that $1/z$ can be approximated by $\partial u/\partial x - i(\partial u/\partial y)$ where u is a real valued function in $C^1(\mathbb{R}^n)$.

In fact $1/z$ can be approximated by linear combinations of other powers of z. Suppose the contrary. Then there exists a measure μ supported on C such that

$$\int_C \frac{d\mu}{z} = 1 \qquad \text{and} \qquad \int_C z^n \, d\mu(z) = 0 \qquad \text{for} \quad n \neq -1.$$

From this it immediately follows that $\int_C d\mu(\xi)/(\xi - z) = 1$ for $z \in \operatorname{int} C$ and $= 0$ for $z \in \operatorname{ext} C$ (exterior of C). Under these circumstances, C must be rectifiable! (Garnett brought this to my notice as one of Wermer's theorems. Here we give a natural proof of this fact and I am sure it is the same as Wermer's.)

Let φ be the Riemann mapping from $|z| < 1$ to $\operatorname{int} C$ such that $\varphi(0) = 0$. We define a measure ν on $|z| = 1$ such that

$$\int_{|z|=1} f \cdot \varphi(z) \, d\nu(x) = \int_C f(\xi) \, d\mu(\xi).$$

It is clear that ν exists and is unique. By hypothesis $d\mu$ is orthogonal to all polynomials and hence all mappings that are holomorphic on int C and continuous on C. Hence

$$\int_C (\varphi^{-1}(\xi))^n \, d\mu(\xi) = 0 \qquad \text{for all } n \geqslant 0$$

and consequently

$$\int z^n \, d\nu(z) = \qquad \text{for} \quad n \geqslant 0.$$

Therefore by F and M Riesz' theorem, $d\nu = K(z)\, dz$ where

$$\int_{|z|=1} |K| |dz| < \infty$$

and K is holomorphic in the unit disc. But since

$$\int_C \frac{d\mu(\xi)}{\xi - \xi_0} = 1 \qquad \text{for all } \xi_0 \in \text{int } C,$$

we have

$$\int_{|z|=1} \frac{K(z)}{\varphi(z) - \varphi(z_0)} \, dz = 1 \qquad \text{for} \quad |z_0| < 1.$$

which implies that $K(z_0) = 2\pi i \varphi'(z_0)$. Hence $\int_{|z|=1} |\varphi'(z)| \, |dz| < \infty$ which means C is rectifiable.

Theorems 2, 3 and 4 have the same conclusion under a different hypothesis and so let us state the conclusion. Given any $\epsilon > 0$ and $V: R^n \to \mathbb{R}^n$ a continuous mapping, there exists a $u \in C^1(\mathbb{R}^n)$ such that $|V - \nabla u| < \epsilon$ on the set E. Henceforward we shall state only the hypothesis on E in the enunciation of our theorems.

THEOREM 2. *E is totally disconnected.*

Proof. There exists a $\delta > 0$ such that for $x, y \in V$ and $|x - y| < \delta$, $|V(x) - V(y)| < \epsilon$.

Furthermore, there would exist x_i $(1 \leqslant i \leqslant N)$ such that $B(x_i\,; \delta)$ $(v \leqslant i \leqslant N)$ would cover E. Since E is totally disconnected, there would exist mutually disjoint open sets U_j $(1 \leqslant j \leqslant M)$ covering E and each U_j is contained in $B(x_i\,; \delta)$ for some i. Let us select a $\tau(j)$ so that $U_j \subset B(x_{\tau(j)}\,; \delta)$.

Now let us define $h(x) = x \cdot v(x_{\tau(j)})$ in U_j. If $U = \bigcup_{j=1}^{M} U_j$, h is well defined on U and harmonic on U and

$$|V - \nabla h| < \epsilon \quad \text{on} \quad E.$$

Remark. Since the complement of E is connected, we can replace our h by a suitable harmonic polynomial.

THEOREM 3. $\alpha: [0, 1] \to \mathbb{R}^n$ *is a Jordan arc and* $E = |\alpha| =$ *support of* α.

LEMMA. *Let* $N \geqslant 2$; x_i $(0 \leqslant i \leqslant N)$ *be* $N + 1$ *distinct points in* $\mathbb{R}^n$; c_i $(0 \leqslant i \leqslant N)$ *be any* $N + 1$ *vectors in* $\mathbb{R}^n$. *Given any* $\epsilon > 0$, *we can find vectors* ϵ_i $(0 \leqslant i \leqslant N)$, $\epsilon_0 = \epsilon_N = 0$, $|\epsilon_i| < \epsilon$ *such that* $b_i - b_{i+1}$ *is not orthogonal to the vector* $\overline{x_i x_{i+1}}$ *where* $b_i = c_i + \epsilon_i$.

There is nothing to the proof of the lemma. We may remark that $N \geqslant 2$ is essential since we do not want to alter c_0 and c_N.

Proof of Theorem 3. Let us select a large integer N with the following properties: (i) $N \geqslant 2$; (ii) $|V(\alpha(t)) - V(\alpha(t'))| < \epsilon$ for $|t - t'| < 1/N$. Let x_i denote $\alpha(i/N)$, c_i denote $V(\alpha(i/N))$ for $0 \leqslant i \leqslant N$. By the lemma there exists b_i such that $|b_i - c_i| < \epsilon$, $b_i - b_{i+1}$ is not orthogonal to $\overline{x_i x_{i+1}}$. Let π_i denote the plane passing through the midpoint of $\overline{x_i x_{i+1}}$ and normal to $b_i - b_{i+1}$ for $0 \leqslant i \leqslant N - 1$. By our construction π_i separates x_i from x_{i+1}.

Now it is clear that there exist neighborhoods U_i of $\alpha[t_i, t_{i+1}]$ such that

(1) $U_i \cap U_j = \varnothing$ for $|i - j| > 1$;

(2) $|V(x) - V(\alpha(t_i))| = |V(x) - c_i| < \epsilon$ for $x \in U_i$ and consequently $|V(x) - b_i| < 2\epsilon$ for $x \in U_i$; and

(3) π_{i+1} separates $\overline{U_i \cap U_{i+1}}$ from $\overline{U_{i+1} \cap U_{i+2}}$ for $0 \leqslant i \leqslant N - 2$.

Let us define $h(x) = x \cdot b_0$ whenever $x \in U_0$ and lies on the same side of π_0 as x_0. And for $x \in U_0$ and lying on the opposite side of π_0, i.e., the side of x_1, we define $x \cdot b_1 + k_1$ where k_1 is so chosen that $x \cdot b_0$, $x \cdot b_1 + k_1$ coincide on $\pi_0 \cap U_0$. Our construction of π_0 is such that this is possible.

We move from here to U_1. For $x \in U_1$ and lying on the same side of π_1 as x_1, we define $h(x) = x \cdot b_1 + k_1$ and for $x \in U_1$ and lying on the same side of π_1 as x_2, we define $h(x) = x \cdot b_2 + k_2$ where k_2 is so chosen that $x \cdot b_1 + k_1 = x \cdot b_2 + k_2$ on $U_1 \cap \pi_1$. It is possible since π_1 is normal to $b_1 - b_2$. We also see that h agrees with its definition on U_0. It is now clear how to continue $h(x)$ to the whole of $U = \bigcup_{i=0}^{N-1} U_i$.

The function h thus defined is continuous on U and piecewise linear and further $|V - \nabla h| < 2\epsilon$. Now it only remains to smooth our h to complete the proof but that is quite standard.

THEOREM 4. $\alpha: [0, 1] \to R^n$ *is a* non-rectifiable *Jordan curve.* $E = |\alpha|$.

Proof. Let us select an integer $N \geqslant 3$ such that $|V(\alpha(t)) - V(\alpha(t'))| < \epsilon$ for $|t - t'| < 1/N$. One of the arcs $\alpha[i/N, (i + 1)/N]$ is nonrectifiable and there is no less of generality if we assume $\alpha[(N-1)/N, 1]$ is nonrectifiable. As before let us define $x_i = \alpha(i/N)$; $c_i = V(\alpha(i/N))$ for $0 \leqslant i \leqslant N-2$ and $c_{n-1} = c_0$. Applying the lemma of Theorem 3, we get hold of b_i such that $b_0 = c_0$, $b_{N-1} = c_{N-1}$, and $| b_i - c_i | < \epsilon$ for $0 \leqslant i \leqslant N-1$ and also that $b_i - b_{i+1}$ is not perpendicular to $\overline{x_i x_{i+1}}$ for $0 \leqslant i \leqslant N-2$. Hence by the same method as in the previous theorem we construct a piecewise linear continuous function h in a neighborhood U of $\alpha[0, (N-1)/N]$ such that $| V - \nabla h | < 3\epsilon$ and further $h = x \cdot c_0$ in a neighborhood of x_0 and $= x \cdot c_0 + k$(const) in a neighborhood of x_{N-1}.

Hence if we can prove the existence of a C^∞-function u with $| \nabla u | < \epsilon$ on $\alpha[(N-1)/N, 1]$ and $u \equiv k$ in a neighborhood of x_{N-1} and $u = 0$ in a neighborhood of $x_N = x_0$, we are done.

Thus we are reduced to the following:

Proposition. *Given* α: $[0, 1] \to \mathbb{R}^n$ *a* nonrectifiable *Jordan arc and given any* $\epsilon > 0$, *there exists a* C^∞*-function* u *in* R^n *such that* $| \nabla u | < \epsilon$ *on* $| \alpha |$, $u \equiv 1$ *in a neighborhood of* $\alpha(0)$ *and* $u \equiv 0$ *in a neighborhood of* $\alpha(1)$.

Proof of the Proposition. Let N be so large that

$$\sum_{r=0}^{N-1} \left| \alpha\left(\frac{i}{N}\right) - \alpha\left(\frac{i+1}{N}\right) \right| > \frac{2}{\epsilon}.$$

Let x_i denote $\alpha(i/N)$. If three consecutive points x_i are colinear, we will drop the middle one. After doing so, we will reach a stage where we cannot drop any more. Then we will be left with $t_0 = 0 < t_1 < \cdots < t_N = 1$ (this N could be less than the previous one) such that of the points $\alpha(t_i)$, no three consecutive points are colinear. Now let again x_i denote $\alpha(t_i)$ and a_i denote the unit vector in the direction $\overline{x_{i+1} x_i}$ for $0 \leqslant i \leqslant N-1$. It is clear that

$$\sum_{i=0}^{N-1} | x_i - x_{i+1} | > \frac{2}{\epsilon}.$$

Let y_i be any interior point of the segment $x_i x_{i+1}$ to be fixed later. Let π_0 be the plane passing through y_0 and normal to a_0 and let π_i be the plane passing through y_i and normal to $a_i - a_{i-1}$ for $1 \leqslant i \leqslant N-1$. Clearly π_i separates x_i from x_{i+1}. Hence there exist neighborhoods U_i of $\alpha[t_i, t_{i+1}]$ $(0 \leqslant i \leqslant N-1)$ such that

(1) $\bar{U}_i \cap \bar{U}_j = \varnothing$ for $| i - j | > 1$; and
(2) π_i separates $\overline{U_{i-1} \cap U_{i+1}}$ from $\overline{U_i \cap U_{i+1}}$ for $i > 0$.

Let us define $h(x)$ on $U = \bigcup_{i=1}^{N-1} U_i$. Let us start with U_0. For $x \in U_0$ and lying on the same side of π_0 as x_0, we define $h(x) \equiv 1$ and for x lying on the opposite side in U_0, we define $h(x) = (x - y_0) \cdot \epsilon a_0 + 1$. We observe that $h(x_1) = (x_1 - y_0) \epsilon a_0 + 1$. For $x \in U_1$ and lying on the same side of π_1 as x_1, define $h(x) = (x - y_0) \cdot \epsilon a_0 + 1$ and on the opposite side Thus $h(x) = (x - y_1) \cdot \epsilon a_1 + (y_1 - y_0) \cdot \epsilon a_0 + 1$. There is no dash on $\pi_1 \cap U_1$. Thus $h(x_2) = (x_2 - y_1) \cdot \epsilon a_1 + (y_1 - y_0) \cdot \epsilon a_0 + 1$. We have done it before and we can do the same thing again, namely, define h inductively on U. Here we observe that

$$h(x_N) = (x_N - y_{N-1}) \cdot \epsilon a_{N-1} + \sum_{i=0}^{N-2} (y_{i+1} - y_i) \cdot \epsilon a_i + 1.$$

Hence as $y_i \to x_i$, $h(x_N) \to -\epsilon \sum_{i=0}^{N-1} | x_i - x_{i+1} | + 1$ which is less than -1.

Hence we can select a y_N close to x_N on the segment $x_{N-1}x_N$ and plane π_N passing through y_N and orthogonal to $x_{N-1}x_N$ and redefine $h(x) \equiv h(y_N)$ on that part of U_{N-1} which lies on the same side of π_N as x_N. Thus we are able to define a piecewise linear h on U such that $| \nabla h | < \epsilon$ and $h \equiv 1$ in a neighborhood of $\alpha(0)$ and $h \equiv -k$ in a neighborhood of $\alpha(1)$ where $k > 1$. Let $h_1 = (h + k)/(1 + k)$. Then h_1 satisfies: (i) $h_1 \equiv 0$ in a neighborhood of $\alpha(1)$; (ii) $h_1 \equiv 1$ in a neighborhood of $\alpha(0)$; and (iii) $| \nabla h_1 | < \epsilon$. Smoothing h_1 we prove the proposition. Q.E.D.

Remark. Our general problem in Theorems 3 and 4 was to characterize compact subsets E of $\mathbb{R}^n$ on which any continuous vector-valued function $v: \mathbb{R}^n \to \mathbb{R}^n$ can be approximated by ∇u, $u \in C^\infty(\mathbb{R}^n)$. Dually posed, the problem takes the following form: *Given n measures μ_i $(1 \leqslant i \leqslant n)$, finite and supported on E such that for every vector $\alpha = (\alpha_i) \in \mathbb{R}^n$, $\sum \alpha_i \mu_i$ projected on α in zero; what conditions on E would ensure that each μ_i is separately null.* In the case of totally disconnected sets, we can prove the dual problem directly. In the case of curves, however, the only proof I know is to first prove Theorems 3 and 4 and to conclude the truth of their duals.

The Case of a Rectifiable Jordan Curve

THEOREM 5. *Let α: [0, 1] $\to \mathbb{R}^n$ be a rectifiable Jordan curve and V be any continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $\int_\alpha \sum_{i=1}^n v_i \, dx_i = \int_\alpha v \cdot dx = 0$. Then given any $\epsilon > 0$, there exists a $u \in C^\infty(\mathbb{R}^n)$ such that $| V - \nabla u | < \epsilon$ on $| \alpha |$.*

Outline of Proof. Subtracting grad $v(\alpha(0)) \cdot x$ from V, we may assume $V(\alpha(0)) = 0$. Then there exists a $\delta > 0$ such that for $0 \leqslant t \leqslant \delta$, $| V(\alpha(t))| < \epsilon$. Hence

$$\left| \int_\delta^1 V \cdot \frac{d\alpha}{dt} \, dt \right| = \left| \int_0^\delta V \cdot \frac{d\alpha}{dt} \, dt \right| \leqslant \epsilon$$

arc length of $\alpha[0, \delta] = \alpha_1 \cdot \epsilon$, for instance. Evidently we can select $\delta = t_0 < t_1 < \cdots < t_N = 1$ such that $|\sum_{i=1}^{N-1} V(\alpha(t_i)) \cdot (\alpha(t_i) - \alpha(t_{i+1}))| < 2\epsilon \cdot \alpha_1$, and $|V(\alpha(t)) - V(\alpha(t_i))| < \epsilon$ for $t_i \leqslant t \leqslant t_{i+1}$. Since $|V(\alpha(t_0))|$ and $|V(\alpha(t_N))| = |V(\alpha(0))|$ are $< \epsilon$, exactly as we have done in Theorem 4, we can prove the existence of a piecewise linear continuous function u in a neighborhood of $\alpha[\delta, 1]$ such that $|V - \nabla u| < \epsilon$ and further $u \equiv 0$ in a neighborhood of $\alpha(\delta)$ and $u \equiv$ *a* constant K in a neighborhood of $\alpha(1) = \alpha(0)$.

Since K can be selected as close to $\sum_{i=0}^{N-1} V(\alpha(t_i)) \cdot (\alpha(t_i) - \alpha(t_{i+1}))$ as we please, we may assume $|K| < 2\epsilon \cdot \alpha_1$.

Now our problem is to descend from K to 0 along α from $\alpha(0)$ to $\alpha(\delta)$ together with the condition that $|\nabla u|$ should be small. This is the same problem as one had in the proposition of Theorem 4. Since the length of $\alpha[0, \delta]$ is α_1 and $|K| < 2\epsilon \cdot \alpha$, we can do it (as in the proposition quoted) with $|\nabla u| < 4\epsilon$. Thus we constructed a piecewise linear continuous u in a neighborhood of $|\alpha|$ such that $|V - \nabla u| < 4\epsilon$. Q.E.D.

Remark. In all these approximations u was harmonic except for a set of measure zero, in fact a finite number of plane sections. But of course smoothing certainly decreases the set of points of harmonicity of u. Certainly we came very close to proving what one did in the case of totally disconnected sets, namely u is harmonic in a neighborhood of $|\alpha|$. But unfortunately we could not, without the additional assumption that meas $|\alpha| = 0$, thus appealing to Theorem 1. In the case of the rectifiable Jordan curve, the condition of measure zero is fulfilled.

It would be desirable to prove, in the case of an arbitrary Jordan arc or curve (not necessarily rectifiable) that ∇u, $u \in C^\infty(\mathbb{R}^n)$ can be replaced by ∇H where H is harmonic in a neighborhood of the support of the curve.

References

1. S. N. Mergelyan, Approximate Cauchy problem for the Laplace operator, *Uspekhi Matematičeskih Nauk* **11** (1956), 3–26.